Lesson 07

Classification, Boosting, Support Vector Machine, Decision Trees

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Principles of Classification

- classification vs. clustering - with/without teacher
- **Feature vector**
- is an *n*-dimensional vector describing attributes of the classified object/event
- for the purpose of generality lets assume that a feature vector $x \in \mathcal{R}^n$
- the task of a binary classificator is to divide the $\mathcal{R}^n$ space into two parts so that (ideally) all vectors from one class lie in one part of the space and vice versa
- generally a hyperplane is used as a solution of this problem

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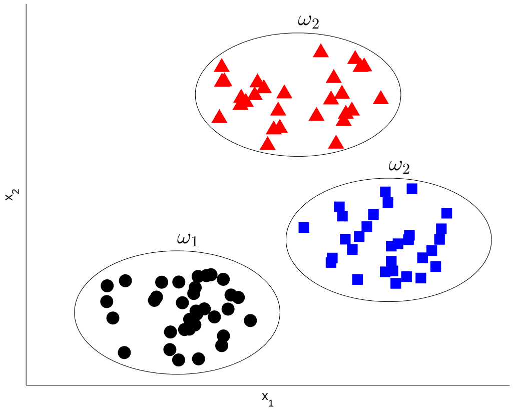
- ω_i is the i^{th} class, $X \in \mathcal{R}^n$ is the space of all classes

$$\begin{aligned} \omega_{1,\dots,N} &\subset X, \\ \bigcup_{i=1}^{N} \omega_i &= X, \\ \omega_i \cap \omega_j &= \emptyset, \text{pro} \quad i,j = 1,\dots,N, \quad i \neq j, \end{aligned}$$



Ada-Boost

- Ada-Boost (short for Adaptive Boosting) is an algorithm creating a strong classifier as a combination of weak classifiers
- a **weak classifier** is such classifier that performs at better than a random choice, i.e. the error $\epsilon < 0.5$ for a binary classification problem
- lets denote a weak classifier as $h(x) \rightarrow \{-1; 1\}$
- a strong classifier is a linear combination of weak classifiers, lets denote it as $H(x) = sign \sum_{t=1}^{T} \alpha_t h_t(x)$

Algorithm

- we have training data available $\left\{\left(x^{(i)}, y^{(i)}\right)\right\}_1^N, y \rightarrow \{-1; 1\}$
- initialize weights corresponding to individual feature vectors as $\omega_0(i) = 1/N$
- for $t = 1, \dots, T$:
- compute $h_t = \operatorname{argmin}_{h_j \in \mathcal{H}} \epsilon_j = \sum_{i=1}^N \omega_i [y_i \neq h_j(x_i)]$
- if $\epsilon_t \geq 0.5$ then stop - the classifier failed to train
- set $\alpha_t = \frac{1}{2} \log \left(\frac{1-\epsilon_t}{\epsilon_t} \right)$
- update $\omega_{t+1}(i) = \omega_t(i) \exp(-\alpha_t y_i h_t(x_i))/Z_t$
- iterate until $\epsilon_t = 0$
- the final strong classifier $H(x) = sign \sum_{t=1}^T \alpha_t h_t(x)$

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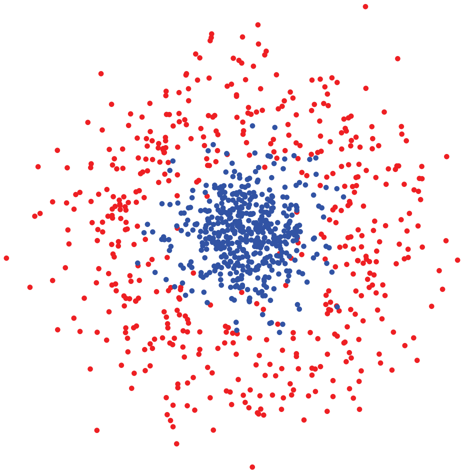

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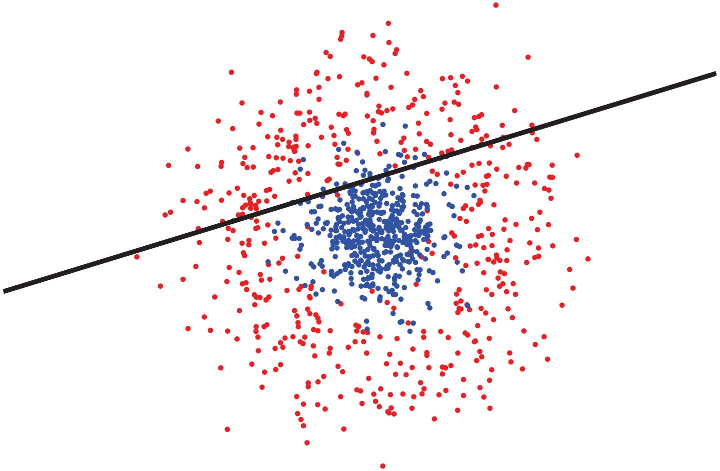

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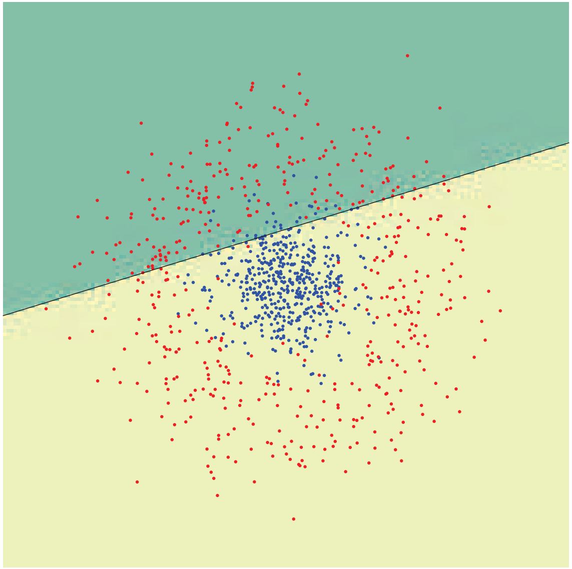

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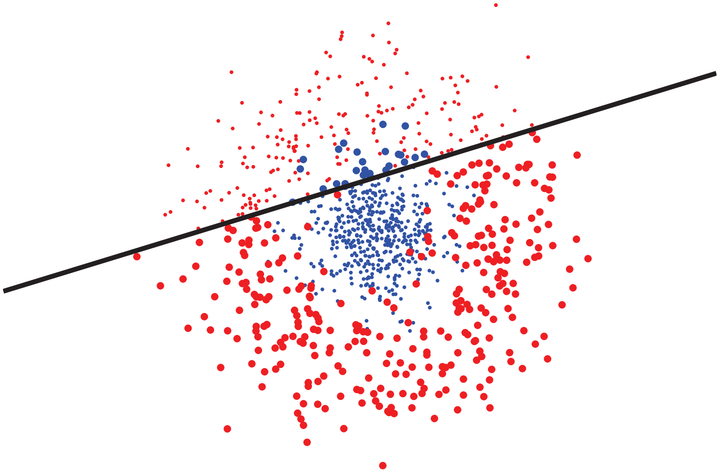

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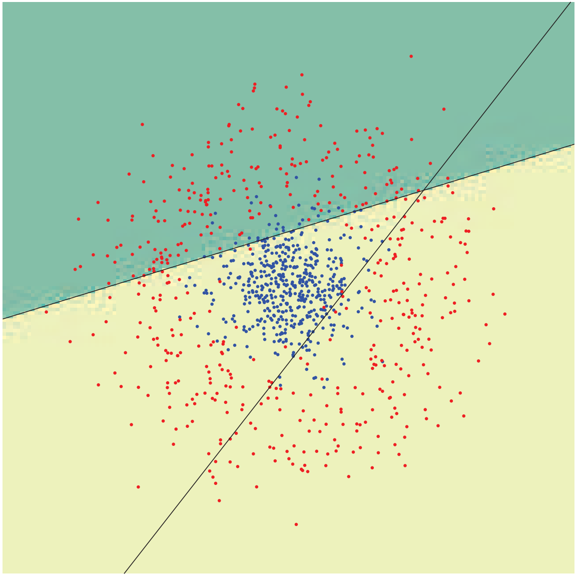

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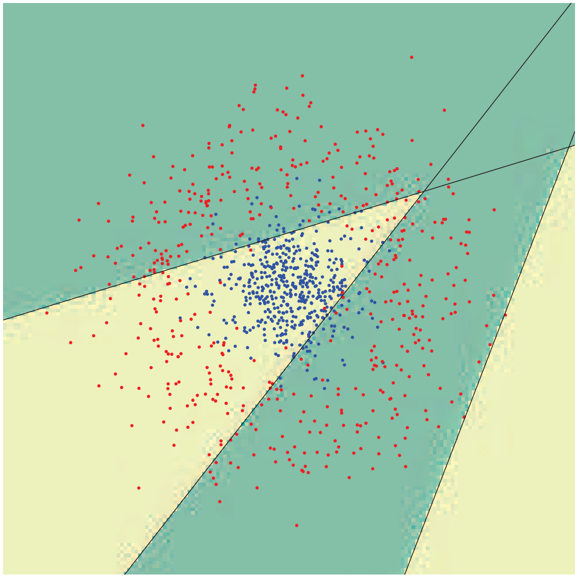

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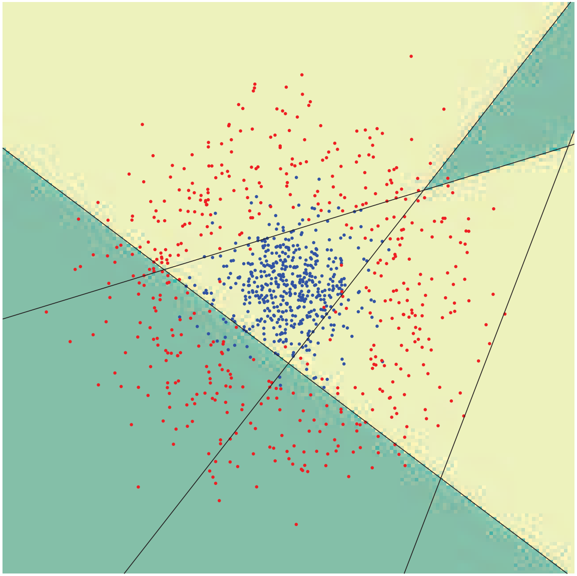

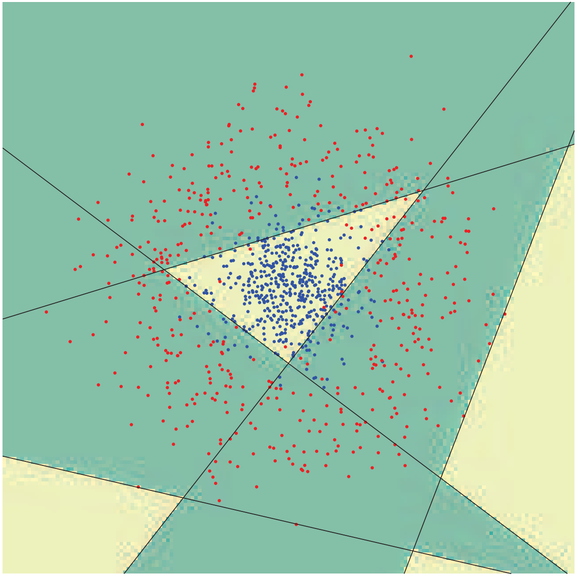

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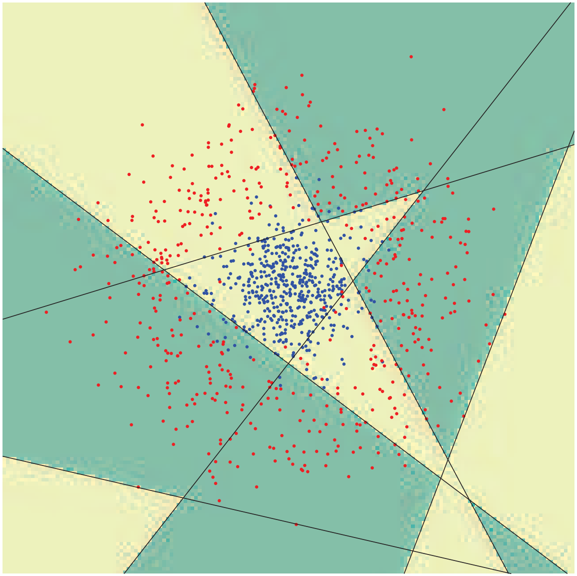

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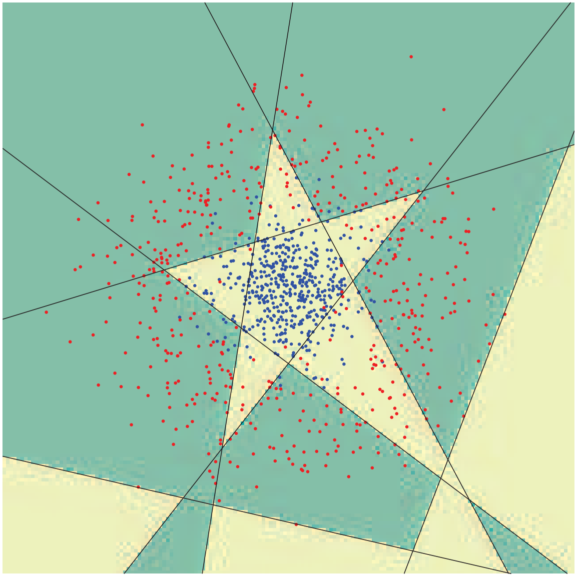

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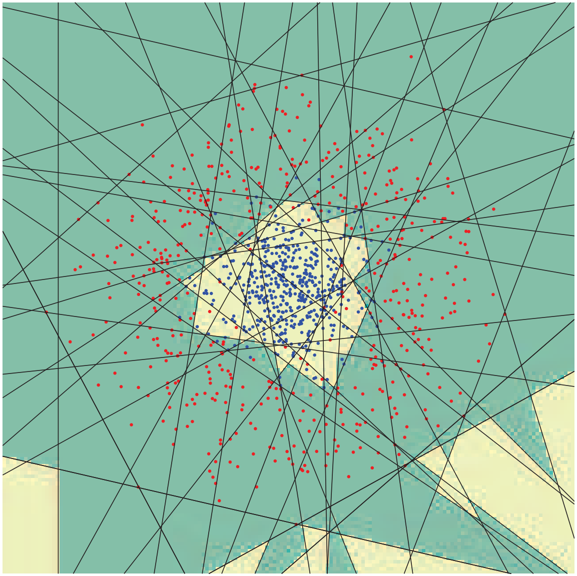

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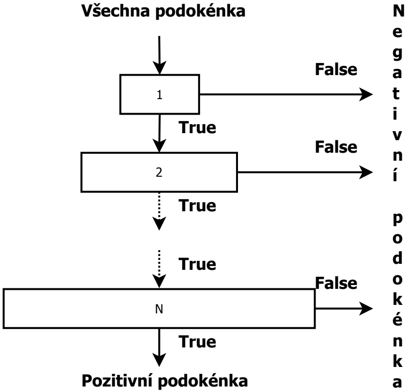
Cascade Ada-Boost

- is a special framework for ada-boost
- the goal is to make the recognition faster but still efficient
- the decision is made sequentially - this allows to refuse some features in very early stages



Support Vector Machine

- in previous sections we have shown how to compute a decision boundary
- in the case of linearly separable classes there exist a lot of boundaries that will classify the training set with 100% precision
- the question is: Is there (in some sense) an optimal decision boundary?

Support Vector Machine

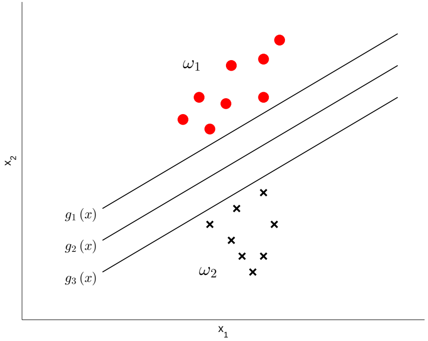
- in previous sections we have shown how to compute a decision boundary
- in the case of linearly separable classes there exist a lot of boundaries that will classify the training set with 100% precision
- the question is: Is there (in some sense) an optimal decision boundary?
- The criterion: The distance between the boundary and the nearest training vector is maximized

- we have a training set $\left\{\left(x^{(i)}, y^{(i)}\right)\right\}_1^N, y \rightarrow \{-1; 1\}$
- we have to find the parameters of a decision boundary $\boldsymbol{\omega}$ (previously $\boldsymbol{\Theta}$)

$$\boldsymbol{\omega}^\top \mathbf{x} > 0, \quad \text{pro } \forall \mathbf{x} \in \omega_1,$$
$$\boldsymbol{\omega}^\top \mathbf{x} < 0, \quad \text{pro } \forall \mathbf{x} \in \omega_2.$$

- the decision boundary is then defined as:

$$g(\mathbf{x}) = \boldsymbol{\omega}^\top \mathbf{x} + \omega_0 = 0, \tag{1}$$



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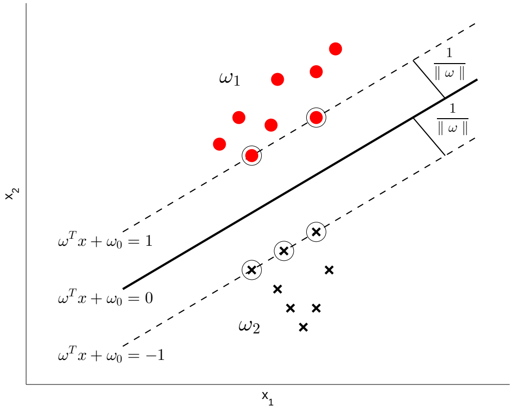
- as said, SVM tries to find the optimal boundary based on the distances from the training data
- with some normalization and math this can be achieved relatively easily



ω_1
ω_2
$\frac{1}{\| \omega \|}$
$\frac{1}{\| \omega \|}$
$\omega^T x + \omega_0 = 1$
$\omega^T x + \omega_0 = 0$
$\omega^T x + \omega_0 = -1$
x_2
x_1

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- we want to find such parameters $\boldsymbol{\omega}$ that will satisfy:

$$\boldsymbol{\omega}^\top \mathbf{x} + \omega_0 \geq +1, \quad \text{pro } \forall \mathbf{x} \in \omega_1,$$
$$\boldsymbol{\omega}^\top \mathbf{x} + \omega_0 \leq -1, \quad \text{pro } \forall \mathbf{x} \in \omega_2.$$

- and we know that the distance between the hyperplanes satisfying the equality in the equations above will be $\frac{2}{\|\boldsymbol{\omega}\|}$
- we want this distance to be maximized
- this leads to the criterion $J = \min \|\boldsymbol{\omega}\|$ which for the math sake will be changed to $J = \min \frac{1}{2}\|\boldsymbol{\omega}\|^{\mathbf{2}}$
- but with the condition of good classification

$$y_i \left(\boldsymbol{\omega}^\top \mathbf{x}_i + \omega_0\right) \geq 1, \quad i = 1, 2, \dots, N. \tag{2}$$

- the vectors $\mathbf{x_i}$ that satisfy $y_i \left(\boldsymbol{\omega}^\top \mathbf{x}_i + \omega_0\right) = 1$ are called support vectors

Optimization of the SVM criterion

- to optimize a criterion with conditions we make use of the Lagrangian multiplicator

$$\mathcal{L}\left(\boldsymbol{\omega}, \omega_0, \boldsymbol{\lambda}\right) = \frac{1}{2}\boldsymbol{\omega}^\top\boldsymbol{\omega} - \sum_{i=1}^{N} \lambda_i \left[y_i \left(\boldsymbol{\omega}^\top \mathbf{x}_i + \omega_0\right) - 1\right] \tag{3}$$

- we need to find the minimum of $\mathcal{L}$
- we use partial derivations

$$\frac{\partial}{\partial \boldsymbol{\omega}}\mathcal{L}\left(\boldsymbol{\omega}, \omega_0, \boldsymbol{\lambda}\right) = 0 \tag{4}$$

$$\frac{\partial}{\partial \omega_0}\mathcal{L}\left(\boldsymbol{\omega}, \omega_0, \boldsymbol{\lambda}\right) = 0 \tag{5}$$

- to optimize a criterion with conditions we make use of the Lagrangian multiplicator

$$\mathcal{L}\left(\boldsymbol{\omega}, \omega_0, \boldsymbol{\lambda}\right) = \frac{1}{2}\boldsymbol{\omega}^\top\boldsymbol{\omega} - \sum_{i=1}^{N} \lambda_i \left[y_i \left(\boldsymbol{\omega}^\top \mathbf{x}_i + \omega_0\right) - 1\right] \tag{6}$$

- this leads to the solution

$$\boldsymbol{\omega} = \sum_{i=1}^{N} \lambda_i y_i \mathbf{x}_i, \tag{7}$$

$$0 = \sum_{i=1}^{N} \lambda_i y_i. \tag{8}$$

Dual form

- we make use of the dual form of the problem
- we take the primal solution and substitute it to the primal problem and find the maximum

$$\min_{\boldsymbol{\omega},\omega_0} \left(\frac{1}{2}\boldsymbol{\omega}^\top \boldsymbol{\omega} - \sum_{i=1}^{N} \lambda_i \left[y_i \left(\boldsymbol{\omega}^\top \mathbf{x}_i + \omega_0 \right) - 1 \right] \right) \tag{9}$$

- becomes

$$\max_{\lambda} \left(\sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i,j} \lambda_i \lambda_j y_i y_j \mathbf{x}_i^\top \mathbf{x}_j \right) \tag{10}$$

- maximizing this equation yields the solution for λ_i which when substituted to the equation $\boldsymbol{\omega} = \sum_{i=1}^{N} \lambda_i y_i \mathbf{x}_i$ give us the solution for ω

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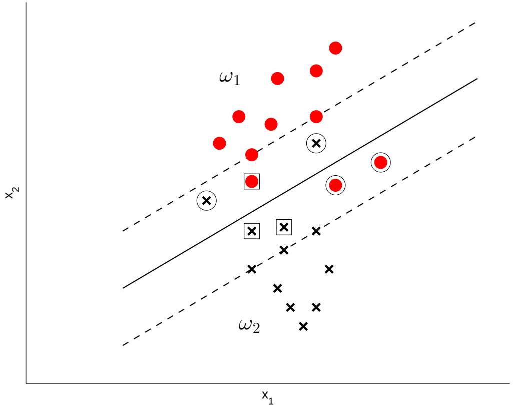
Soft-margin

- when the classes are linearly non-separable



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- vectors that are correctly classified: $y_i \left(\boldsymbol{\omega}^\top \mathbf{x}_i + \omega_0\right) \geq 1$
- vectors that are correctly classified but lie in the margin: $0 \leq y_i \left(\boldsymbol{\omega}^\top \mathbf{x} + \omega_0\right) < 1$
- vectors that are misclassified: $y_i \left(\boldsymbol{\omega}^\top \mathbf{x} + \omega_0\right) < 0$
- this can be written as:

$$y_i \left(\boldsymbol{\omega}^\top \mathbf{x} + \omega_0\right) \geq 1 - \xi_i \tag{11}$$

- the goal is to find the hyperplane that maximizes the margin and minimizes the number of points for which $\xi > 1$
- this leads to a new formulation of the problem:

$$J\left(\boldsymbol{\omega}, \omega_0, \boldsymbol{\xi}\right) = \frac{1}{2} \parallel \boldsymbol{\omega} \parallel^2 + C \sum_{i=1}^{N} I\left(\xi_i\right), \tag{12}$$

$$I\left(\xi_i\right) = \begin{cases} 1, & \xi_i > 0, \\ 0, & \xi_i = 0. \end{cases} \tag{13}$$

Kernel Trick

- in the solution of the SVM:

$$\max_{\lambda} \left(\sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i,j} \lambda_i \lambda_j y_i y_j \mathbf{x}_i^\top \mathbf{x}_j \right) \tag{14}$$

- we can see the dot product of x_i, x_j
- this can be efficiently written with the kernel trick as

$$\max_{\lambda} \left(\sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i,j} \lambda_i \lambda_j y_i y_j K\left(\mathbf{x}_i, \mathbf{x}_j\right) \right) \tag{15}$$

- this represents a transformation of the vectors into a higher dimension
- in this higher dimension the vectors can be linearly separable

Kernel Types

Type of kernel	Formula	Note
Polynomial	$K\left(\mathbf{x}_i, \mathbf{x}_j\right) = \left(\mathbf{x}_i \mathbf{x}_j + \theta\right)^d$	Parameter d and threshold θ is chosen by user.
Sigmoid kernel	$K\left(\mathbf{x}_i, \mathbf{x}_j\right) = tahh\left(\eta \mathbf{x}_i \mathbf{x}_j + \theta\right)$	Parameter η and threshold θ is chosen by user.
Gauss kernel *Radial Basis Function* (RBF)	$K\left(\mathbf{x}_i, \mathbf{x}_j\right) = exp\left(-\frac{1}{2\sigma^2} \parallel \mathbf{x}_i - \mathbf{x}_j \parallel^2\right)$	Parameter σ is is chosen by user.

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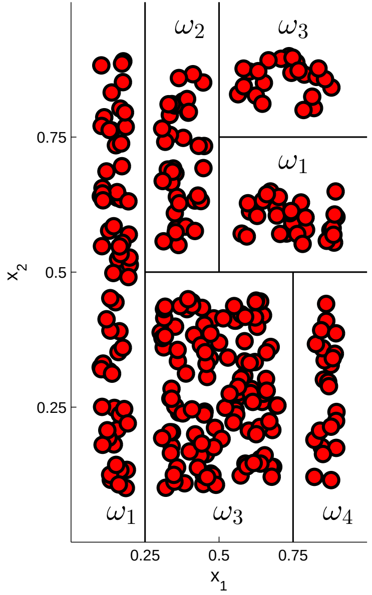
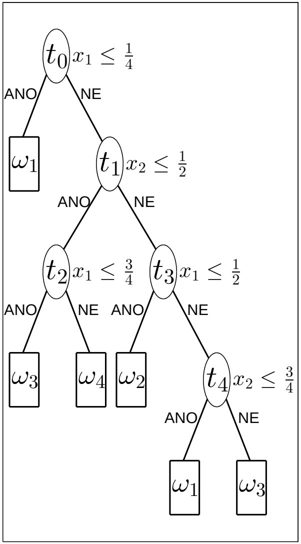
Decision Tree

- non-linear classification method, the model is based on oriented graph $\rightarrow$ tree

Decision Tree

- non-linear classification method, the model is based on oriented graph $\rightarrow$ tree
- belongs to a family of models - Classification And Regression Tree (CART)





t_0 $x_1 \leq \frac{1}{4}$
ANO
NE
ω_1
t_1 $x_2 \leq \frac{1}{2}$
ANO
NE
t_2 $x_1 \leq \frac{3}{4}$
t_3 $x_1 \leq \frac{1}{2}$
ANO
NE
ANO
NE
ω_3
ω_4
ω_2
t_4 $x_2 \leq \frac{3}{4}$
ANO
NE
ω_1
ω_3

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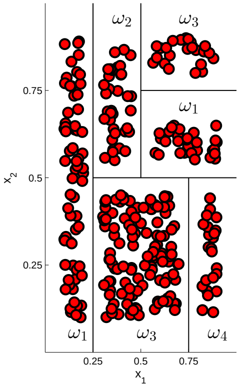
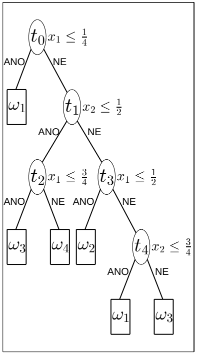
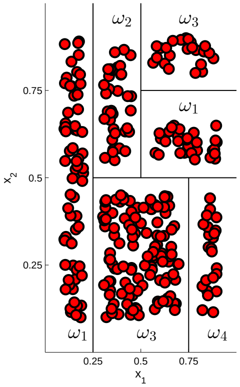
- method uses binary decision tree T consisting of nodes $\rightarrow$ elements of feature vector $\mathbf{x} \in X$ are evaluated via a condition

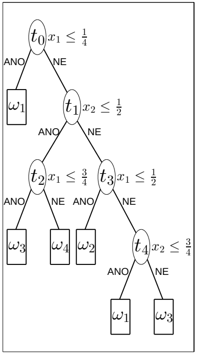




t_0 $x_1 \leq \frac{1}{4}$
ANO
NE
ω_1
t_1 $x_2 \leq \frac{1}{2}$
ANO
NE
t_2 $x_1 \leq \frac{3}{4}$
t_3 $x_1 \leq \frac{1}{2}$
ANO
NE
ANO
NE
ω_3
ω_4
ω_2
t_4 $x_2 \leq \frac{3}{4}$
ANO
NE
ω_1
ω_3

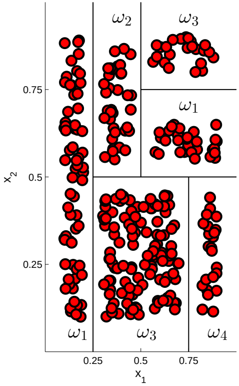
- method uses binary decision tree T consisting of nodes $\rightarrow$ elements of feature vector $\mathbf{x} \in X$ are evaluated via a condition
- the tree then represents a gradual segmentation of the feature space X into disjunct regions

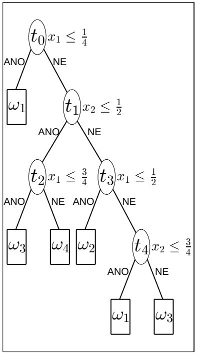




t_0 $x_1 \leq \frac{1}{4}$
ANO
NE
ω_1
t_1 $x_2 \leq \frac{1}{2}$
ANO
NE
t_2 $x_1 \leq \frac{3}{4}$
t_3 $x_1 \leq \frac{1}{2}$
ANO
NE
ANO
NE
ω_3
ω_4
ω_2
t_4 $x_2 \leq \frac{3}{4}$
ANO
NE
ω_1
ω_3

- each region represents one and only one class





t_0 $x_1 \leq \frac{1}{4}$
ANO
NE
ω_1
t_1 $x_2 \leq \frac{1}{2}$
ANO
NE
t_2 $x_1 \leq \frac{3}{4}$
t_3 $x_1 \leq \frac{1}{2}$
ANO
NE
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ω_4
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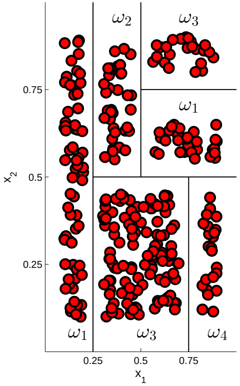
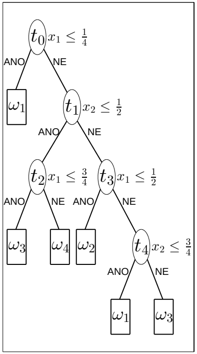
ANO
NE
ω_1
ω_3

- each region represents one and only one class
- the feature space is divided into rectangular regions (the region boundaries are parallel to axis of feature space)





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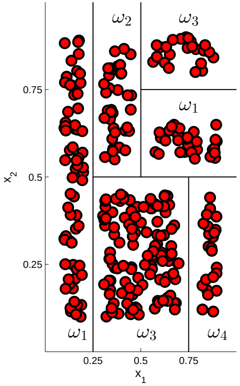
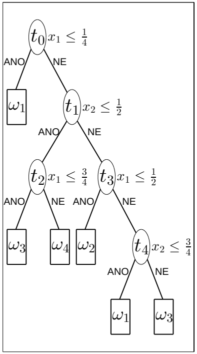
NE
t_2 $x_1 \le \frac{3}{4}$
t_3 $x_1 \le \frac{1}{2}$
ANO
NE
ANO
NE
ω_3
ω_4
ω_2
t_4 $x_2 \le \frac{3}{4}$
ANO
NE
ω_1
ω_3

- each region represents one and only one class
- the feature space is divided into rectangular regions (the region boundaries are parallel to axis of feature space)
- the inequations in nodes $x_i \leq \alpha$ is known as the decision rule





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ANO
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ω_1
t_1 $x_2 \leq \frac{1}{2}$
ANO
NE
t_2 $x_1 \leq \frac{3}{4}$
t_3 $x_1 \leq \frac{1}{2}$
ANO
NE
ANO
NE
ω_3
ω_4
ω_2
t_4 $x_2 \leq \frac{3}{4}$
ANO
NE
ω_1
ω_3

Classification and Learning:

- classification proceeds by comparing an unknown vector in the nodes of the tree
- the unknown vector then falls into one of the leafs which represents a class
- usually the learning is supervised (learning with teacher)
- straightforward way of training $\rightarrow$ the regions are constructed by comparing values in individual dimensions of the vector $\mathbf{x}$ with a threshold, $x_i \leq \alpha$, x_i is the i^{th} element of the feature vector $\mathbf{x}$ and α is a threshold

Rules of the construction of the decision tree:

- the first node (root) of the tree contains the whole training set, $X_s = X$
- every next node s contains the subset $X_s \subset X$ given by the decision rule of the previous node
- the decision rule divides X_s into two subsets X_{sT} (TRUE) and X_{sF} (FALSE)
- the division must fulfill:

$$X_{sT} \cap X_{sF} = \emptyset,$$
$$X_{sT} \cup X_{sF} = X_s.$$

- from all the possible divisions of X_s we pick just one, which is optimal given a **division criterion**

On choosing the decision rule

- the decision rule in the form $x_i \leq \alpha_i$, where α_i is a threshold $\alpha_i \in \mathrm{R}$ divides the feature vectors based on the comparison of the i^{th} dimension of the feature vector
- thanks to the train set X it is possible to enumerate a finite set of values for computing α_i
- for the i^{th} dimension of feature space the values of all feature vectors on this dimension are ordered ascending $\rightarrow$ we have a finite set of values for computing the threshold
- in a given node we can enumerate all the possible values from all the dimensions x_i
- from this set of possible divisions (values of the threshold) we need to choose such that will divide the given set of feature vectors "the best"$\rightarrow$ we need a metric (eg. Gini impurity, variance reduction, information gain, ...)

Information gain approach

- let $P(\omega_i \mid s)$ be the probability of vectors in the set X_s belonging to the class ω_i
- the information gain is based around the entropy:

$$I(s) = -\sum_{i=1}^{M} P(\omega_i \mid s) \log_2 P(\omega_i \mid s). \qquad (16)$$

- this equation represents the rate of *entropy* of the node s
- the probabilities $P(\omega_i \mid s)$ are estimated by $\frac{N_s^i}{N_s}$, where N_s^i is the number of vectors in X_s belonging to class ω_i and N_s is the total number of vectors in the subset X_s

- after dividing X_s into two subsets X_{sT} a X_{sF}, where X_{sT} is composed of N_{sT} vectors and X_{sF} is composed of N_{sF} vectors, the information gain (of this division) is:

$$\Delta I(s) = I(s) - \frac{N_{sA}}{N_s} I(s_A) - \frac{N_{sN}}{N_s} I(s_N), \qquad (17)$$

  where $I(s_A)$, $I(s_N)$ are the rates of entropy of nodes s_A and s_N
- the goal of the training is to find for each node s such division for which the information gain $\Delta I(s)$ is maximized

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Stopping criterion

- is used to stop the process of division and thus creating a leaf node
- one option is to set the minimal number of training vectors in the node
- another option is to set a minimal information gain that is needed for the division

Classification

- a leaf node s represents the class for which there are the most training vectors in the leaf node
- each leaf node represents one class ω_j, where j is

$$j = \operatorname*{argmax}_i P\left(\omega_i \mid s\right). \tag{18}$$

Other options of constructing the tree:

- the decision rule can have the form of $\sum_{i=1}^{l} c_i x_i \leq \alpha$
- we are not looking for thresholds but for parameters of a hyperplane that divides the feature space into two subsets
- when considering two dimensional feature space and by rearranging the expression we obtain: $c_1 x + c_2 y - \alpha \leq 0$
- which is a general form of equation of a half-plane
- can be more suitable in some cases, but the construction of the tree is more complex

Decision forest

- a disadvantage of the decision tree is the sensitivity to the training set, so called bad generalization
- a small change in training set X results in change of topology of the whole decision tree T
- this drawback is compensated by using more trees in the training/testing phase
- principle: for one training set we construct several different trees

Training:

- the training set X is divided into several training sets $X_{(t)}$ by utilizing the *bootstrap aggregating* algorithm

Classification:

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- unknown vector $\mathbf{y}$ is inputed into all decision trees
- each decision tree outputs the class ω_i for the unknown vector $\mathbf{y}$
- index i of the final class is chosen as the most frequent result, alternatively we may compute the probability for each class as $P(\omega_i|\mathbf{y}) = \frac{1}{T}\sum_{t=1}^{T} P_t(\omega_i|\mathbf{y})$

Náhodný rozhodovací les (Random Decision Forest)

- the same principle as the decision forest $\rightarrow$ lowering the sensitivity of classification on the training set
- . . . but also
- goal 1: lowering the correlation of the trees in the forest
- goal 2: make the training faster (especially for higher dimensions)

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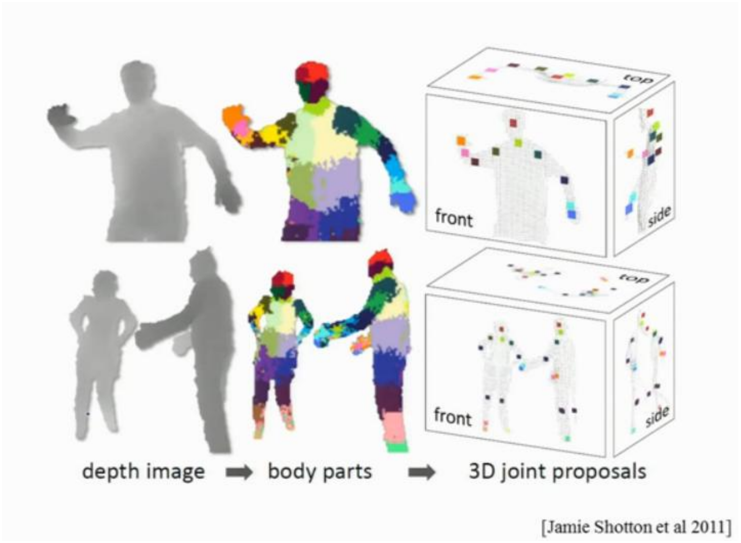
Real-time classification of depth data from MS Kinect into individual parts of human body (Microsoft Research, 2011):



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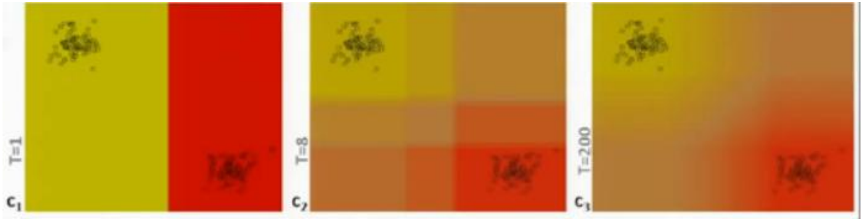
Training

1. **division of the training set** X into T sets $X_{(t)}$ using *bootstrap aggregating* (the same)
2. we choose a parameter m ($m \ll l$, where l is the dimensionality of $\mathbf{x} \in X$)
3. for one tree in a given node $\leftarrow$ the decision rule is determined based only on randomly chosen m dimensions
4. after the tree is trained, choose another m dimensions and train another tree, and so on

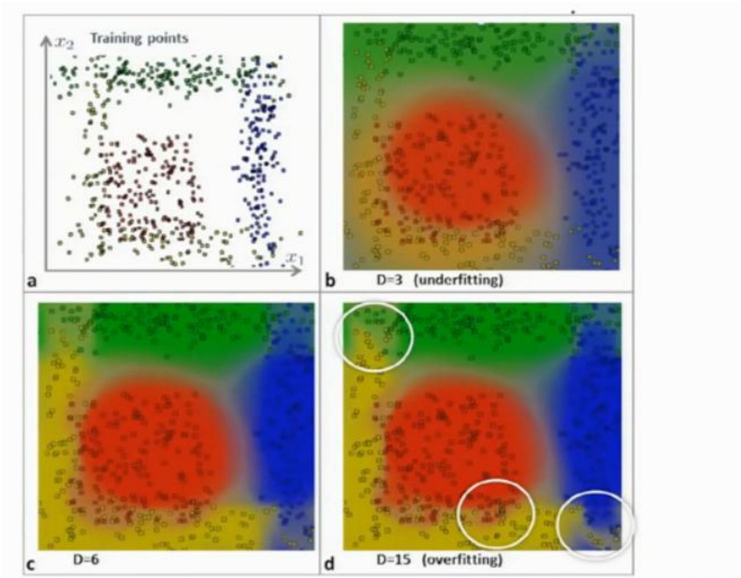
Classification

- an unknown vector $\mathbf{y}$ is inputed into all the trees
- index i of the final class is chosen as the most frequent result, alternatively we may compute the probability for each class as $P(\omega_i|\mathbf{y}) = \frac{1}{T}\sum_{t=1}^{T} P_t(\omega_i|\mathbf{y})$

Effect of the size of the forest:



Effect of the depth of the trees:



[Criminisi et al. 2011]